

On the structure of the constraint algebra for systems whose gauge transformations depend on higher order time derivatives of the gauge parameters

M.N. Stoilov

*Bulgarian Academy of Sciences,
Institute of Nuclear Research and Nuclear Energy,
Blvd. Tzarigradsko Chaussee 72, Sofia 1784, Bulgaria
e-mail: mstoilov@inrne.bas.bg*

February 2, 2008

Abstract

The dynamical systems invariant under gauge transformations with higher order time derivatives of the gauge parameter are considered from the Hamiltonian point of view. We investigate the consequences of the basic requirements that the constraints on the one hand and the Hamiltonian and constraints on the other hand form two closed algebras. It is demonstrated that these simple algebraic requirements lead to rigid relations in the constraint algebra.

key words: higher stage constraints

Introduction

Dynamical systems in which the gauge transformations involve higher order derivatives of the gauge parameters are considered in the literature both from the Lagrangean and Hamiltonian point of view [1]–[4]. In the Lagrangean approach the corresponding Noether identities are obtained and in the Hamiltonian approach the constraints generating the gauge transformations are constructed. In the present paper we address some aspects of the possible representations of the constraint algebra in the Hamiltonian approach.

Any dynamical system with gauge symmetry is characterized in the Hamiltonian approach by its Hamiltonian H and constraints φ_a , $a = 1, \dots, c$. The Hamiltonian and constraints are functions of the phase space variables q_m and p_m , $m = 1, \dots, n$. The constraints generate the gauge transformation of any phase space dynamical quantity $g(p, q)$ through the Poisson bracket relations

$$\delta_\epsilon g = \epsilon_a [g, \varphi_a]. \quad (1)$$

The Hamiltonian generate (again through the Poisson bracket relations) the time evolution (up to a gauge transformation) of any g

$$\dot{g} \equiv \frac{dg}{dt} = [g, H]. \quad (2)$$

In eq.(1) the parameters of the gauge transformation ϵ_a can be arbitrary functions of the time t . Due to the specific character of the time in the Hamiltonian approach there are no time derivatives of any order of ϵ_a in the transformation (1). On the other hand, if we consider a field theory it is possible to have spatial derivatives acting on ϵ_a .

It is not possible to pick up arbitrary Hamiltonian and constraints and to obtain a well defined dynamical model. There are some consistency conditions which the Hamiltonian and constraints have to satisfy. First, the commutator of two gauge transformations has to be a gauge transformation. Together with the Jacobi identity this means that constraints form a closed gauge algebra with respect to the Poisson bracket relations

$$[\varphi_a, \varphi_b] = C_{abe} \varphi_e. \quad (3)$$

Here C_{abc} are the structure functions of the gauge algebra. Second, the time evolution has to preserve the gauge algebra (3). In other words the Hamiltonian and constraints also have to form a closed algebra, i.e. besides eq.(3) the following relation has to be satisfied as well

$$[H, \varphi_a] = U_{ab} \varphi_b. \quad (4)$$

In the simplest but very common case the structure functions C_{abe} and U_{ab} do not depend on the dynamical variables. In this case both the gauge algebra (3) and the algebra of the constraints and Hamiltonian (3,4) are Lie algebras. If the gauge algebra is semi-simple or Abelian then the structure constants C_{abe} only matters for the algebra of the Hamiltonian and constraints. The structure constants U_{ab} are not important because they are due to weakly zero terms (terms proportional to the constraints) in the Hamiltonian [5]. Such terms can be freely removed from the Hamiltonian (thus obtaining the so called ‘canonical Hamiltonian’) and if we do so, we get that U_{ab} are zeros. In other words, the canonical Hamiltonian is always gauge invariant.

The Hamiltonian approach to the constraint systems is equivalent to the first order Lagrangean approach with the following Lagrangean.

$$L = p\dot{q} - H - \lambda_a \varphi_a. \quad (5)$$

Here λ_a are the Lagrange multipliers. Their gauge transformation is given below:

$$\delta_\epsilon \lambda_a = \partial_t \epsilon_a + \epsilon_c C_{cba} \lambda_b. \quad (6)$$

Note that we have a term with time derivative of the gauge parameters in eq.(6). It has been already stressed that the Hamiltonian approach does not allow time derivatives of the gauge parameter. Therefore, we need some modification of this approach if we want to handle within it the gauge transformation of the Lagrange multipliers. It is shown in Ref.[5] that the transformation (6) can be generated by the following constraints which act in the phase space of Lagrange multipliers λ_a and their momenta π_a

$$\hat{\varphi}_a = \overleftarrow{\partial}_t \pi_a + \lambda_b C_{abc} \pi_c. \quad (7)$$

Here we introduce the operator of the time derivative $\overleftarrow{\partial}_t$ which acts on the gauge parameters and not on the phase space variables. If we do not use the canonical Hamiltonian then in eqs.(6,7) some extra terms proportional to U_{ab} appear.

Eq.(6) is an example of a gauge transformation with first order time derivative of the gauge parameter. This example gives us grounds to ask the question is it possible, e.g. in the second order Lagrangean formalism, to have dynamical variables whose gauge transformation involves higher than first order time derivatives of the gauge parameter? The answer of this question is positive. The aim of the present paper is to investigate the algebra of constraints which generates gauge transformations with higher time derivatives of the gauge parameter. Some aspects of this problem are considered in [1],[2]. Here we focus our attention on the consequences of the required

Lie algebraic structure. In our investigation we use eq.(7) as a pattern: the constraints $\hat{\varphi}_a$ are polynomials with respect to the time derivative operator $\overleftarrow{\partial}_t$ with coefficients functions in a specific phase space. We expect the same structure for the generators of the gauge transformations involving higher order time derivatives of the gauge parameters. Loosely speaking we shall refer to such gauge transformations as ‘higher stage’ ones.

Higher stage gauge transformations

An example

There is a simple example with higher stage transformations of any finite order. Consider a mechanical model with n coordinates q_m , $m = 1, \dots, n$ with the following Lagrangean

$$L = \frac{1}{2} \sum_{m=2}^n (\dot{q}_{m-1} - q_m)^2. \quad (8)$$

The model is invariant with respect to the following one parametric gauge transformation:

$$\delta_\epsilon q_m = \partial_t^{m-1} \epsilon. \quad (9)$$

We recall that the parameter ϵ can be arbitrary function of the time. The Dirac analysis of the Lagrangean (8) shows that we have a primary constraint ($p_n = 0$), a secondary constraint ($p_{n-1} = 0$) and so on up to n -th stage constraint ($p_1 = 0$). All of these constraints are first class. On the base of this analysis we expect an n -parametric gauge symmetry, but the symmetry (9) is only one parametric. Therefore, none of the primary, secondary and so on constraints do not generate independent gauge symmetry. These constraints are projection of the unique gauge symmetry generator in different subspaces of the phase space — $\{q_n, p_n\}$, $\{q_{n-1}, p_{n-1}\}$ and so on. An interesting feature of the considered model is that the gauge parameter in the different subspaces is not the same. As it is seen from eq.(9) the parameter of the gauge transformation in the subspace $\{q_m, p_m\}$ is $\partial_t^{m-1} \epsilon$. An analogy with eqs.(6,7) suggests that we have to use the operator of the time derivative acting on the gauge parameter when we write down the constraint generating the transformation (9). The generator which we are looking for is:

$$\psi = \sum_{m=1}^n \overleftarrow{\partial}_t^{m-1} p_m \quad (10)$$

where $\overleftarrow{\partial}_t^i$ is the i -th time derivative acting on the gauge parameter ϵ . Note that the coefficients in this series are the different stage constraints which we obtain through the Dirac prescription.

The general construction

Hereafter we shall consider only models with finite highest stage gauge transformations. Without this condition the model will be non-local in time. The general form of the finite higher stage gauge variation is

$$\delta_\epsilon q_m = \epsilon_a \sum_{i=k_0}^k \frac{\overleftarrow{\partial}_t^i}{i!} f_{am}^i. \quad (11)$$

Here k_0 and k are the minimal and maximal order of the gauge parameter time derivatives and f_{am}^i are some (yet unspecified) functions. Without any loss of generality we can accept that k_0 is zero because the case in which $k_0 \neq 0$ can be brought to the case $k_0 = 0$ with a redefinition of the parameters ϵ_a , such that $\epsilon^{new} = \partial_t^{k_0} \epsilon$.

If the Lagrangean of the model is not with higher derivatives then f_{am}^i are functions of q and \dot{q} and the symmetry (11) can be realized in the phase space of the model [2]. Here we adopt a slightly different approach. Having in mind eq.(7), we are looking for a realization of the higher stage gauge transformation (11) in some larger phase space with coordinates $\{q, p\}$. This phase space contains besides the initial phase space of the model also the phase space of the Lagrangean multipliers, additional phase space variables connected with (possible) higher derivatives and second class constraints, ghosts, etc.

In general, the enlargement of the phase space requires redefinition of the constraints and the Hamiltonian. Terms which live in the new dimensions have to be added both to the initial Hamiltonian and constraints so that the gauge algebra and the algebra of the Hamiltonian and constraints to remain the same. However, if we are using the canonical Hamiltonian there is no need to modify it. In other words, the canonical Hamiltonian in the enlarged phase space is a function of the initial phase space variables only. The reason is the gauge invariance of the canonical Hamiltonian. On the other hand, if we for some reasons do not use the canonical Hamiltonian a procedure like the construction of the BRST invariant Hamiltonian has to be carried out. Here we assume that we are working with the canonical Hamiltonian. Therefore, the only things we have to find in the enlarged phase space are the constraints. We are looking for the generators ψ of the transformation

(11) in the following form:

$$\psi_a = \sum_{i=0}^k \frac{\overleftarrow{\partial}_t^i}{i!} \varphi_a^i \quad (12)$$

where the different stage constraints φ_a^i are such functions in the enlarged phase space so that for any $g(\underline{q}, \underline{p})$

$$\delta_\epsilon g = \epsilon_a [g, \psi_a]. \quad (13)$$

(In the above equation $[,]$ denotes the Poisson brackets in the $\{\underline{q}, \underline{p}\}$ phase space.)

Consistency conditions

The basic requirements that the gauge generators on the one hand and the Hamiltonian and gauge generators on the other hand must form closed algebras are valid for any gauge model including the models with higher stage gauge transformations. Therefore, for the higher stage gauge generators the following relation must hold

$$[\psi_a, \psi_b] = C_{abc} \psi_c. \quad (14)$$

Hereafter we suppose that the algebra (14) is a Lie algebra which we shall denote \mathcal{A} . The requirement that the time evolution of the constraints does not produce new constraints leads to the gauge invariance of the canonical Hamiltonian [5] in the case of 0-stage gauge transformations which form Abelian or semi-simple Lie algebra. The result however does not depend on the particular realization of the gauge algebra. So, even for the higher stage gauge transformations the canonical Hamiltonian has to be gauge invariant, i.e.

$$\epsilon_a [H, \psi_a] = 0. \quad (15)$$

From the above equation we get using the arbitrariness of the gauge parameters ϵ_a that

$$[H, \varphi_a^i] = 0 \quad \forall a, i. \quad (16)$$

From eqs.(12, 14) and after a series expansion on different powers of $\overleftarrow{\partial}_t$ we get the Poisson bracket relations between the different stage generators φ_a^i :

$$[\varphi_a^i, \varphi_b^j] = \theta_{i+j}^k C_{abc} \varphi_c^{i+j}. \quad (17)$$

Here θ_j^i is the step symbol

$$\theta_j^i = \begin{cases} 1 & \text{if } i \geq j \\ 0 & \text{if } j > i \end{cases} \quad (18)$$

Using the fact that C_{abc} are structure constants of a Lie algebra it is easy to check that the set $\{\varphi_a^i\}$ generates a Lie algebra as well. This algebra we denote \mathcal{B}^k . It follows from eq.(17) that \mathcal{A} is a sub-algebra of \mathcal{B}^k . The algebra \mathcal{B}^k has the following Killing form:

$$g_{ai \, bj}^{\mathcal{B}} = \delta_{i0} \delta_{j0} g_{ab}^{\mathcal{A}} \quad (19)$$

where $g_{ab}^{\mathcal{A}}$ is the Killing form of the algebra \mathcal{A} . Eq.(19) leads to the following Levi-Malcev decomposition of \mathcal{B}^k in the case when \mathcal{A} is semi-simple

$$\mathcal{B}^k = \mathcal{C} + \mathcal{A}. \quad (20)$$

In the above semi-direct sum decomposition the algebra \mathcal{C} is generated by φ_a^i with $i > 0$ while \mathcal{A} is generated by φ_a^0 . It turns out that the algebra \mathcal{C} is not only solvable but it is nilpotent.

Some representations of the algebra \mathcal{B}^k

Matter representation

Suppose we know a representation $\pi(\mathcal{A})$ of the algebra \mathcal{A} acting in a d -dimensional vector space V . Then it is possible to construct a representation $\Pi(\mathcal{B}^k)$ of \mathcal{B}^k in the $(k+1).d$ -dimensional space $\bigoplus_{k+1} V = \underbrace{V \oplus V \oplus \dots \oplus V}_{k+1}$.

Let us denote by A_a the $d \times d$ matrix representing ψ_a (or φ_a^0)

$$A_a = \pi(\psi_a) \quad (21)$$

The representation $\Pi(\mathcal{B}^k)$ is given by block matrices such that

$$\Pi(\varphi_a^m)_{ij} = \theta_j^k \delta_j^{k+i} A_a. \quad (22)$$

In eq.(22) the subscripts $i, j = 0, \dots, k$ indicate the block row and column position and the block contents is always the matrix A_a . In other words, the Π representation of φ_a^0 is given by a block diagonal matrix with the matrix A_a in every diagonal block, and all other blocks equal to zero; $\Pi(\varphi_a^1)$ is given by a matrix for which in any block along the block diagonal above the main block diagonal sits the matrix A_a , all other blocks zero, and so on till $\Pi(\varphi_a^k)$

for which the only non-zero block is in the upper right corner where again the matrix A_a sits.

The above construction can be realized in a phase space with coordinates $\{q_u^0, \dots, q_v^k, p_u^0, \dots, p_v^k\}$ where $u, v = 1, \dots, d$ as follows:

$$\Pi_a^m = - \sum_{i=0}^{k-m} q^i A_a p^{i+m} \quad (23)$$

In this realization q^0 transforms as a vector (matter), q^1 transforms as \dot{q}^0 , while the gauge transformations of the other coordinates are more complicated:

$$\delta_\epsilon q^i = - \sum_{j=0}^i \frac{1}{j!} \epsilon_a^{(j)} q^{i-j} A_a. \quad (24)$$

Connection representation

If the representation $\pi(\mathcal{A})$ is the adjoint one (and so, $V = \mathcal{A}$) then it is possible to construct a representation of the algebra \mathcal{B}^k in a smaller space, namely in $\bigoplus^k V$. Let the matrices $\bar{\Pi}_a^0, \dots, \bar{\Pi}_b^{k-1}$ realize a representation of the algebra \mathcal{B}^{k-1} in $\bigoplus^k V$ as described in eq.(22). Let T_a^i , $i = 0, \dots, k-1$ are the translation generators in $\bigoplus^k V$. The meaning of the indices i and a of T_a^i is as follows: the index i indicates the space in the direct sum and the index a indicates the coordinate in this space on which the generator T_a^i acts. Note that the dimension of the adjoint representation is c (the number of constraints) so the range of the translation indices a is correct. The translation generators satisfy the following commutation relations

$$\begin{aligned} [T_a^i, T_b^j] &= 0 \quad \forall i, j \text{ \& } \forall a, b \\ [\bar{\Pi}_a^{k-1}, T_b^j] &= \theta_{i+j}^{k-1} C_{abe} T_e^{i+j}. \end{aligned} \quad (25)$$

We are looking for linear combinations $\bar{\Pi}_a^i$ of the operators T_a^i and $\bar{\Pi}_a^{k-1}$.

$$\begin{aligned} \bar{\Pi}_a^0 &= \bar{\Pi}_a^{k-1} \\ \bar{\Pi}_a^i &= \bar{\Pi}_a^{k-1} + \alpha_i T_a^{i-1}, \quad i = 0, \dots, k-1 \\ \bar{\Pi}_a^k &= \alpha_k T_a^{k-1} \end{aligned} \quad (26)$$

such that $\bar{\Pi}_a^i$, $i = 0, \dots, k$ to satisfy the commutator relations of the algebra \mathcal{B}^k . The result is that the coefficients α_i have to be such that

$$\alpha_{i+j} = \alpha_i + \alpha_j \quad i + j \leq k \quad (27)$$

The solution of the system (27) we shall use is

$$\alpha_i = i. \quad (28)$$

The dynamical realization of the above construction is in a phase space with coordinates $\{q_a^0, \dots, q_b^{k-1}, p_a^0, \dots, p_b^{k-1}\}$ ($a, b = 1, \dots, c$). The gauge transformation of the coordinate q_a^i in this case is:

$$\delta_\epsilon q_a^i = - \sum_{j=0}^i \frac{1}{j!} \epsilon_e^{(j)} q_b^{i-j} C_{eba} + \frac{1}{i!} \epsilon_a^{(i+1)} \quad (29)$$

As it seen from the above equations, the coordinates q_a^0 transform as connection and q^1 transforms as \dot{q}^0 . Note that the gauge transformation of the Lagrange multipliers (6) is of this type.

The Lagrangean with higher stage gauge symmetry

Having a Hamiltonian H and constraints ψ_a it is possible to write down the following Lagrangean by analogy with the Lagrangean (5):

$$L = \underline{p}\dot{\underline{q}} - H - \lambda_a \psi_a. \quad (30)$$

In this Lagrangean the operators of the time derivatives which are part of the definition of the constraints ψ_a act on the Lagrangean multipliers λ_a . It turns out that the Lagrangean (30) is invariant under the higher stage gauge transformation (13) provided

$$\delta_\epsilon \lambda_a = \partial_t \epsilon_a + \epsilon_c C_{cba} \lambda_b. \quad (31)$$

But this is exactly the gauge transformation of the Lagrange multipliers in the standard 0-stage case (see eq.(6)), i.e. the gauge transformation of the Lagrange multipliers does not depend on the stage of the gauge transformation. Therefore, the part of the constraints which acts in the Lagrange multipliers phase space has an universal character and is given by eq.(7). This fact allows us to separate the contribution of the Lagrange multipliers in the Lagrangean (30). Let \hat{q} and \hat{p} denote all phase space variables \underline{q} and \underline{p} but λ and π . Then the Lagrangean (30) can be rewritten in the following form

$$\begin{aligned} L &= \hat{p}\dot{\hat{q}} + \pi\dot{\lambda} - H - \lambda_a \hat{\psi}_a - \lambda_a (\overleftarrow{\partial}_t \pi_a + \lambda_b C_{abc} \pi_c) \\ &= \hat{p}\dot{\hat{q}} - H - \lambda_a \hat{\psi}_a \end{aligned} \quad (32)$$

In eq.(32) $\hat{\psi}_a$ are the generators of the gauge symmetry in the phase space $\{\hat{q}, \hat{p}\}$. Note first, that there is no dependence on π_a in L , so the Lagrange multipliers are purely non-dynamical (as they should be). Second, eq.(32) describes a higher derivative model with gauge freedom. The Hamiltonian approach to such models can be found in Ref.[6].

Conclusion

Eq. (17) shows that simple algebraic requirements lead to very strong relations between the constraints of different stage φ_a^i . The structure of these constraints, as it is seen from eqs.(22,26), is dictated entirely from a representation π of the constraint algebra \mathcal{A} and a number k .

Finally, we want to say few words about the applicability of our results in the field models with gauge freedom. In these models the gauge parameters are functions not only of the time but of the spatial coordinates as well. Therefore, the gauge transformation may depend on (higher) spatial derivatives of the gauge parameter. In this case we can apply the procedure described above for the higher time derivatives to the spatial derivatives. However, there is an essential difference between higher time derivatives and higher spatial derivatives in the Hamiltonian approach — the gauge transformations which depend on the gauge parameter spatial derivatives are handled without any problem in the Hamiltonian approach. Nevertheless, an analysis of the constraints in the spirit of eqs.(12,17) and the reveal of the corresponding algebraic structure (22,26) seems instructive.

References

- [1] Kh.S. Nirov, BRST formalism for systems with higher order derivatives of gauge parameters, Int.J.Mod.Phys. A11 (1996) 5279
- [2] A.A. Deriglazov and K. Evdokimov, Int. J. Mod. Phys. A15 (2000) 4045.
- [3] J.M. Pons, J. Antonio Garcia, Rigid and gauge Noether symmetries for constrained systems, Int.J.Mod.Phys. A15 (2000) 4681-4721
- [4] A. A. Deriglazov, Search for gauge symmetry generators of singular Lagrangian theory, hep-th/0509222
- [5] M.N. Stoilov, Duality between constraints and gauge conditions, Ann. Phys. (Leipzig) 16 (2007) p.529-542.

- [6] D.M. Gitman and I.V. Tyutin, Quantization of Fields with Constraints
(Springer-Verlag, Berlin, 1990)